3.3. Linear Combinations

1. Let V be a vector space, and let S be a subset of V. Any vector of the form
$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_k \mathbf{x}_k,$$
where $\alpha_i \in \mathbb{R}$ and $\mathbf{x}_i \in S$ is called a **linear combination** of the vectors in S.
2. For example, the vector
$$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5 \cdot \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -13 \\ 29 \end{pmatrix}$$
is a linear combination of the set $S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right\}$.

Linear Combinations

Theorem 3.20: A column vector $\vec{\mathbf{x}} \in \mathbb{R}^m$ is a linear combination of column vectors $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n \in \mathbb{R}^m$ if and only if the linear system corresponding to the augmented matrix
$$(\vec{\mathbf{x}}_1 \; \vec{\mathbf{x}}_2 \; \cdots \; \vec{\mathbf{x}}_n \,|\, \vec{\mathbf{x}})$$
has a solution. If the linear system has a solution $(\alpha_1, \ldots, \alpha_n)$, then
$$\alpha_1 \vec{\mathbf{x}}_1 + \alpha_2 \vec{\mathbf{x}}_2 + \cdots + \alpha_n \vec{\mathbf{x}}_n = \vec{\mathbf{x}}.$$

The Span of a Set of Vectors

- Let S be a nonempty subset of a vector space V. We define $\text{span}(S)$ to be the set of all linear combinations of vectors in S. We define the span of the empty set by $\text{span}(\emptyset) = \{\mathbf{0}\}$.
- **Theorem 3.24:** Let S be a subset of a vector space V. Then $\text{span}(S)$ is a subspace of V.
- Notice that the question "Is $\mathbf{x}$ a linear combination of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$?" is equivalent to the question "Is $\mathbf{x}$ in $\text{span}(\{\mathbf{x}_1, \ldots, \mathbf{x}_k\})$?"
- **Theorem 3.26** gives two basic properties of the span of a set of vectors.

Spanning Sets

- Let V be a vector space. If S is a subset of V such that $\text{span}(S) = V$, we say that S is a **spanning set** for V, or that S **spans** V.
- **Theorem 3.28:** Let $S = \{\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n\}$ be a set of column vectors in $\mathbb{R}^m$.
  1. S is a spanning set for $\mathbb{R}^m$ if and only if the rref of the matrix $A = (\vec{\mathbf{x}}_1 \; \vec{\mathbf{x}}_2 \; \cdots \; \vec{\mathbf{x}}_n)$ has a leading 1 in every row. Equivalently, S spans $\mathbb{R}^m$ if and only if $\text{rref}\,(A)$ has no row consisting entirely of zeros.
  2. If $n < m$, then S cannot span $\mathbb{R}^m$.

Linear Dependence and Independence

- Let S be a subset of a vector space V. Then S is a **linearly dependent** set if there exist vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in S$ and scalars $\alpha_1, \ldots, \alpha_k$, not all zero, such that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_k\mathbf{x}_k = \mathbf{0}.$$

  If S is not a linearly dependent set, S is called **linearly independent**.
- Determine if

$$\{\langle 2, 2, -1\rangle, \langle 1, 4, 2\rangle, \langle 5, 6, -2\rangle, \langle 1, 7, 2\rangle\} \subset \mathbb{R}^3$$

  is a linearly dependent set. If so, find a non-trival linear dependence among the vectors.

Linear Dependence and Independence in $\mathbb{R}^m$

Theorem 3.30: Let $S = \{\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k\}$ be a set of column vectors in $\mathbb{R}^m$.

1. S is linearly independent if and only if $\operatorname{rref}\left((\vec{\mathbf{x}}_1\ \vec{\mathbf{x}}_2\ \ldots\ \vec{\mathbf{x}}_k)\right)$ has no free columns.
2. If $k > m$, then S is linearly dependent.

Properties of Linear Dependence and Independence

Theorem 3.32:

1. Let S be a subset of a vector space V. The following are equivalent:
   a. S is linearly dependent
   b. There is a vector $\mathbf{x} \in S$ such that $\mathbf{x} \in \operatorname{span}(S \setminus \{\mathbf{x}\})$.
   c. There is a vector $\mathbf{x} \in S$ such that $\operatorname{span}(S) = \operatorname{span}(S \setminus \{\mathbf{x}\})$.
2. Let $S \subseteq T$ be subsets of a vector space V. If S is linearly dependent, then T is linearly dependent.
3. Let $S \subseteq T$ be subsets of a vector space V. If T is linearly independent, then S is linearly independent.